Faculty of Engineering (Shoubra) Benha University Time allowed: 3 hours	2nd year Production Mechanical Department	Engineering Mathematics and Physics Department 16th of January 2011

The Examination consists of 4 questions with 8 sub questions (10 points for each sub question)

Question 1

a- Evaluate the following integrals:

i) $\int_{0}^{1} t^{aq-1}(1-t^q)^{b-1} dt$

ii) $\int_{0}^{\infty} \frac{e^{2t}-e^{-3t}}{t} dt$

iii) $\int_{0}^{\infty} \frac{a\,dt}{\sqrt{t}(a^2+t^2)}$

b- Find Laplace transform for the following functions:

i) $f(t) = t\sin 2t\cosh 3t + e^{-3t}\int_{u=0}^{t} e^{-2u}\sin u\, du$, ii) $g(t) = \begin{cases} 6 & 0<t\le 2 \\ 3t & 2<t\le 3 \\ 9 & t>3 \end{cases} + \sin^2 4t\, U(t-2)$

Question 2

a- Find inverse Laplace for the following functions:

i) $F(s) = \dfrac{s\,e^{-2s}}{s^2+6s+20} + \dfrac{1}{(s+9)^3}$, ii) $G(s) = \dfrac{1}{s^2(s^2+4)} + \dfrac{3}{s^2+6s+5}$

b) Expand in Fourier series the following functions:

$f(x) = \begin{cases} \pi/2 + x, & -\pi \le x \le 0 \\ \pi/2 - x, & 0 < x \le \pi \end{cases}$

$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ -(x-\pi), & \pi < x \le 2\pi \end{cases}$

Question 3

a-i) Solve the integral equation $\int_{0}^{\infty} f(x)\sin(\alpha x)\, dx = \begin{cases} 1 & 0 \le \alpha < 1 \\ 2 & 1 \le \alpha < 2 \\ 0 & 2 \le \alpha \end{cases}$

ii) Solve the following system of differential equations:

$$\frac{dx}{dt} + 5x + 4y = 10, \quad x - \frac{dy}{dt} - y = 0, \ X(0) = 2, \ Y(0) = -4$$

b-i) Find the constants of the following curve $y(x) = \dfrac{1}{a+bx+cx^2}$ that fit the following data: (-1,2), (3,4), (6,9) using least square method.

ii) Solve using Picard's method up to 2nd approximation the following differential equation:
$y`` = x^3(y`+y)$ , $y(0)=1$ , $y`(0)=½$, then find $y(0.2)$ using Euler method, $h=0.1$.

Question 4

a-i) Find v(x,y), such that f(z) = u(x,y) + i v(x,y) is analytic, where u(x,y)= e^x(xcosy-ysiny). Express f(z) in terms of z.

ii) Find one of the roots of the interpolating Polynomial satisfy the following data (1,3) , (5,-7) , (-13,4) , (2,47) , (-6,15) using inverse Lagrange interpolation.

b- Evaluate the following integrals:

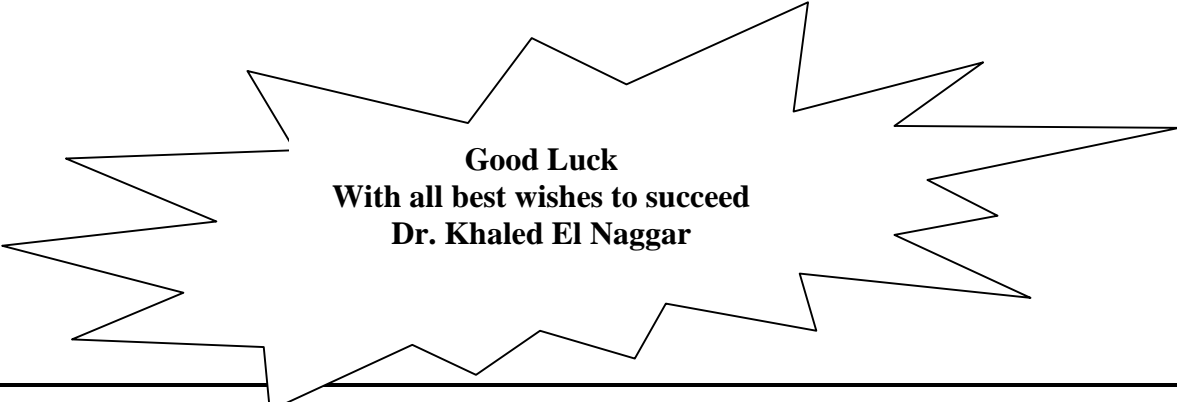i) $\int\limits_{0}^{1/2} t^{m-1} \ln(1/2t)\, dt$,

ii) $\oint\limits_{c} \dfrac{\cos(z)}{z^2 - 6z + 5}\, dz$ where c is the circle $z = 4$.

iii) $\int\limits_{2}^{\infty} e^{-x^2+4x-4}\, dx$,

iv) $\oint\limits_{c} \dfrac{z^3 + 5z + 7}{(z-i)^2}\, dz$, c: $z-2 + z+2 = 6$

v) $\oint\limits_{c} \dfrac{dz}{(z^2 + 9)^2}$, using Cauchy's residue theorem c: $z+1-i = 3$.

Good Luck
With all best wishes to succeed
Dr. Khaled El Naggar

Model answer

Answer of Question 1

a- i) Put $x = t^q$, therefore $dt = (1/q)\, x^{\frac{1}{q}-1} dx$, thus $\int_0^1 t^{aq-1}(1-t^q)^{b-1} dt = \int_0^1 x^{-1/q} x^a (1-x)^{b-1} \frac{1}{q} x^{(1/q)-1} dx$

$= \frac{1}{q}\int_0^1 x^{a-1}(1-x)^{b-1} dx = \frac{1}{q}\beta(a,\,b)$ (4 marks)

ii) Since $L\{\dfrac{e^{2t}-e^{-3t}}{t}\} = \int\limits_{s=0}^{\infty} \dfrac{1}{s-2} - \dfrac{1}{s+3} ds = \text{Ln }(s+3) - \text{Ln }(s-2)$, therefore

$\int\limits_0^{\infty} \dfrac{e^{2t}-e^{-3t}}{t} dt = \text{Ln }(s+3) - \text{Ln }(s-2) \Big|_{s=0} = \text{Ln}(3/2)$ (3 marks)

iii) Put $y = t^2$, therefore $dt = (1/2)\, y^{-\frac{1}{2}} dy$, therefore $\int\limits_0^{\infty} \dfrac{adt}{\sqrt{t}(a^2+t^2)} = \dfrac{a}{2}\int\limits_0^{\infty} \dfrac{y^{-1/2} dy}{y^{1/4}(a^2+y)} =$

$\dfrac{a}{2}\int\limits_0^{\infty} \dfrac{y^{-3/4} dy}{(a^2+y)} = \dfrac{a}{2}\beta\,(1/4,\,3/4) = \dfrac{a}{\sqrt{2}}\pi$ (3 marks)

b-i) $L\{\text{tsin}2t\} = -\dfrac{d}{ds}(\dfrac{2}{S^2+4}) = \dfrac{4s}{(s^2+4)^2}$, therefore $L\{\text{tsin}2t\text{cosh}3t\} = L\{\text{tsin}2t\,(\dfrac{e^{3t}+e^{-3t}}{2})\}$

$= 2[\dfrac{(s-3)}{((s-3)^2+4)^2} + \dfrac{(s+3)}{((s+3)^2+4)^2}]$ (3 marks)

$L\{\int\limits_{u=0}^{t} e^{-2u}\sin u\, du\} = \dfrac{1}{s[(s+2)^2+1]}$, therefore $L\{e^{-3t}\int\limits_{u=0}^{t} e^{-2u}\sin u\, du\} = \dfrac{1}{(s+3)[(s+5)^2+1]}$ (2 marks)

ii) $L\{ \begin{cases} 6 & 0<t\leq 2 \\ 3t & 2<t\leq 3 \\ 9 & t>3 \end{cases} \} = 6[U(t) - U(t-2)] + 3t[U(t-2) - U(t-3)] + 9\, U(t-3)$

$= 6[U(t) - U(t-2)] + 3(t-2+2)\, U(t-2) - 3(t-3+3)U(t-3) + 9\, U(t-3)$

$= 6U(t) + 3(t-2)\, U(t-2) - 3(t-3)\, U(t-3) = 6/s + (3/s^2)\, (e^{-2s} - e^{-3s})$ (3 marks)

$$L\{\sin^2 4tU(t-2)\} = L\ \{\ (\tfrac{1}{2})(1-\cos 8t)\ U(t-2)\} = L\{(\tfrac{1}{2})(1-\cos 8(t-2+2))\ U(t-2)\}$$

$$= (\tfrac{1}{2})\ L\{U(t-2)\} - (\tfrac{1}{2})\ L\{\ [\cos 8(t-2)\cos(16) - \sin 8(t-2)\sin(16)]\ U(t-2)\}$$

$$= (\tfrac{1}{2})e^{-2s} - (\tfrac{1}{2})[\cos(16)\ \frac{s}{(s^2+64)} - \sin(16)\ \frac{8}{(s^2+64)}\ ]\ e^{-2s} \qquad \text{(3 marks)}$$

Answer of Question 2

a-i) Since $F(s) = \dfrac{s\,e^{-2s}}{s^2+6s+20} + \dfrac{1}{(s+9)^3} = \dfrac{(s+3-3)\,e^{-2s}}{(s+3)^2+11} + \dfrac{1}{(s+9)^3}$

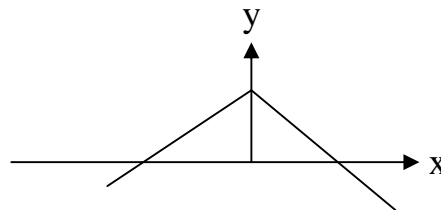$$= [\frac{(s+3)}{(s+3)^2+11} - \frac{3}{(s+3)^2+11}]e^{-2s} + \frac{1}{(s+9)^3}\ , \ \text{therefore}$$

$$f(t)\ = e^{-3(t-2)}\ [\cos\sqrt{11}\ (t-2) - 3/\sqrt{11}\ \sin\sqrt{11}\ (t-2)]\ U(t-2)+(1/2)\ t^2\ e^{-9t} \qquad \text{(5 marks)}$$

ii) $L^{-1}\{\ \dfrac{1}{s(s^2+4)}\ \} = (1/2)\ \displaystyle\int_{u=0}^{t}\ \sin 2u\ du = (1/4)(1-\cos 2t)$, therefore

$$L^{-1}\{\ \frac{1}{s^2(s^2+4)}\ \} = (1/4)\ \int_{u=0}^{t}\ (1-\cos 2u)\ du\ = (1/4)[t-\sin(2t)/2] \qquad \text{(3 marks)}$$

$$L^{-1}\{\ \frac{3}{s^2+6s+5}\ \} = L^{-1}\{\ \frac{3}{(s+3)^2-4}\ \} = (3/2)\ e^{-3t}\sinh(2t) \qquad \text{(2 marks)}$$

b-i) The function is even, therefore
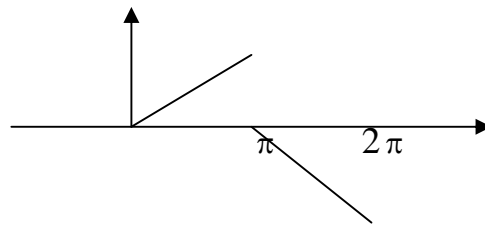

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi}[(\pi/2)-x]\,dx = 0\,,$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi}[(\pi/2)-x]\cos(nx)\,dx = \frac{2}{\pi}[[(\pi/2)-x](\frac{1}{n}\sin(nx))+(\frac{-1}{n^2}\cos(nx)\,)]_{0}^{\pi} = \frac{2}{n^2\pi}(1-\cos n\pi)$$

Therefore $a_{2n}=0$ and $a_{2n-1} = \dfrac{4}{(2n-1)^2\pi}$, thus $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\cos(2n-1)x$ \qquad (5 marks)

4

ii) The function is odd harmonic, therefore

$$a_{2n-1} = \frac{2}{\pi}\int_0^\pi x\cos(2n-1)x\,dx = \frac{2}{\pi}(x(\frac{\sin(2n-1)x}{2n-1})-(\frac{-\cos(2n-1)x}{(2n-1)^2}))\Big|_0^\pi = -\frac{4}{\pi(2n-1)^2}$$

$$b_{2n-1} = \frac{2}{\pi}\int_0^\pi x\,\sin(2n-1)x\,dx = \frac{2}{\pi}(x(\frac{-\cos(2n-1)x}{2n-1})-(\frac{-\sin(2n-1)x}{(2n-1)^2}))\Big|_0^\pi = \frac{2}{(2n-1)}$$

Thus $f(x) = \sum_{n=1}^\infty a_{2n-1}\cos(2n-1)x + \sum_{n=1}^\infty b_{2n-1}\sin(2n-1)x$ (5 marks)

Answer of Question 3

a- i) Since $F_S = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\alpha x)\,dx = \sqrt{\frac{2}{\pi}}\begin{cases}1 & 0\le\alpha<1 \\ 2 & 1\le\alpha<2 \\ 0 & 2\le\alpha\end{cases}$, and Fourier integral is expressed by:

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_S\sin(\alpha x)\,d\alpha = \sqrt{\frac{2}{\pi}}[\int_0^1\sqrt{\frac{2}{\pi}}\sin(\alpha x)\,d\alpha + \int_1^2 2\sqrt{\frac{2}{\pi}}\sin(\alpha x)\,d\alpha] = \frac{2}{\pi x}[1+\cos x - 2\cos 2x]$$

(5 marks)

ii) Take Laplace to the above equations such that:

$s\,X(s) - x(0) + 5X(s) + 3\,Y(s) = 10/s$, $X(s) - [s\,Y(s) - y(0)] - Y(s) = 0$, thus

$(s+5)X(s) + 3\,Y(s) = 10/s + 2$ (1)

$X(s) - (s+1)Y(s) = -3$ (2)

Multiply equation (1) by (s+1) and equation (2) by (3) and add, therefore

$$[s^2+6s+8]\,X(s) = 10(s+1)/s + 2(s+1) + 12$$

Thus $X(s) = \dfrac{10(s+1)}{s(s+4)(s+2)} + \dfrac{2s+14}{s^2+6s+8} = (5/2)(\dfrac{1}{s}-\dfrac{3}{s+4}+\dfrac{2}{s+2}) + \dfrac{2(s+3)+8}{(s+3)^2-1}$

Hence $x(t) = (5/2)(1-3\,e^{-4t}+2e^{-2t}) + e^{-3t}(2\cosh t + 8\sinh t)$, but $\dfrac{dx}{dt}+5x+4y=10$, from which we get

$y(t)$. (5 marks)

b- i) Let $Y = 1/y = a + bx + cx^2$, so that the constants can be obtained using the following matrix form :

$$
\begin{pmatrix}
N & \sum\limits_{i=1}^{N} x_i & \sum\limits_{i=1}^{N} x_i^2 \\
\sum\limits_{i=1}^{N} x_i & \sum\limits_{i=1}^{N} x_i^2 & \sum\limits_{i=1}^{N} x_i^3 \\
\sum\limits_{i=1}^{N} x_i^2 & \sum\limits_{i=1}^{N} x_i^3 & \sum\limits_{i=1}^{N} x_i^4
\end{pmatrix}
\begin{pmatrix} c \\ b \\ a \end{pmatrix}
=
\begin{pmatrix}
\sum\limits_{i=1}^{N} Y_i \\
\sum\limits_{i=1}^{N} x_i Y_i \\
\sum\limits_{i=1}^{N} x_i^2 Y_i
\end{pmatrix}
$$

Given $N = 3$, $\sum\limits_{i=1}^{N} x_i = 8$, $\sum\limits_{i=1}^{N} x_i^2 = 46$, $\sum\limits_{i=1}^{N} x_i^3 = 242$, $\sum\limits_{i=1}^{N} x_i^4 = 1378$, $\sum\limits_{i=1}^{N} Y_i = 0.861$, $\sum\limits_{i=1}^{N} x_i Y_i = 0.92$,

$\sum\limits_{i=1}^{N} x_i^2 Y_i = 6.75$, thus by Cramer rule we get a, b and c such that: $a = \dfrac{\Delta_a}{\Delta}$, $b = \dfrac{\Delta_b}{\Delta}$, $c = \dfrac{\Delta_c}{\Delta}$,

where $\Delta = \begin{vmatrix} 3 & 8 & 46 \\ 8 & 46 & 242 \\ 46 & 242 & 1378 \end{vmatrix}$, $\Delta_a = \begin{vmatrix} 0.86 & 8 & 46 \\ 0.92 & 46 & 242 \\ 6.75 & 242 & 1378 \end{vmatrix}$, $\Delta_b = \begin{vmatrix} 3 & 0.86 & 46 \\ 8 & 0.92 & 242 \\ 46 & 6.75 & 1378 \end{vmatrix}$,

$\Delta_c = \begin{vmatrix} 3 & 8 & 0.86 \\ 8 & 46 & 0.92 \\ 46 & 242 & 6.75 \end{vmatrix}$ (5 marks)

ii) Put $y` = z$, hence $y`` = z` = x^3 (z + y) = f(x, y, z)$, such that $x_0 = 0$, $y_0 = 1$, $y`(0) = z_0 = \tfrac{1}{2}$.

According to the following two formulas, we can get the first two approximations

$$
y_{n+1} = y_0 + \int_{x_0}^{x} z_n \, dx \;, \quad z_{n+1} = z_0 + \int_{x_0}^{x} f(x, y_n, z_n) \, dx = \int_{x_0}^{x} x^3 (y_n + z_n) \, dx
$$

Put n=0 to obtain y_1, z_1 such that $y_1 = y_0 + \int\limits_{x_0}^{x} z_0 \, dx = 1 + \int\limits_{0}^{x} \dfrac{dx}{2} = 1 + \dfrac{x}{2}$ and

$$
z_1 = z_0 + \int_{x_0}^{x} f(x, y_0, z_0) \, dx = \frac{1}{2} + \int_{x_0}^{x} x^3 (y_0 + z_0) \, dx = \frac{1}{2} + \int_0^x x^3 (1 + \frac{1}{2}) \, dx = \frac{1}{2} + \frac{3x^4}{8}
$$

Put n=1 to obtain y_2, such that $y_2 = y_0 + \int\limits_{x_0}^{x} z_1 \, dx = 1 + \int\limits_0^x (\dfrac{3x^4}{8} + \dfrac{1}{2}) \, dx = 1 + \dfrac{x}{2} + \dfrac{3x^5}{40}$ (3marks)

By using Euler method, states:

6

$y_{n+1} = y_n + h\, z_n$, $h = 0.1$ and $z_{n+1} = z_n + 0.1\ x_n^3\ (y_n + z_n)$

$y_1 = y_0 + h\, z_0 = 1 + 0.1(1/2) = 1.05$ and $z_1 = z_0 + 0.1\ x_0^3\ (y_0 + z_0) = \tfrac{1}{2}$

$y_2 = y_1 + h\, z_1 = 1.05 + 0.1(1/2) = 1.1 = y(0.2)$ $\hspace{3cm}$ (2 marks)

Answer of Question 4

a-i) $\dfrac{\partial u}{\partial x} = e^x\,(x\,\cos y + \cos y - y\,\sin y)$ and $\dfrac{\partial u}{\partial y} = e^x(-x\,\sin y - \sin y - y\cos y)$

Since the function is analytic, therefore Cauchy-Riemann equation is satisfied, hence

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = e^x\,(x\,\cos y + \cos y - y\,\sin y)$

Integrate the above equation with respect to y, keeping x constant, then

$v = e^x(x\,\sin y + y\cos y) + \phi(x)$, where $\phi(x)$ is an arbitrary real function of x.

But $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$, therefore $e^x(x\,\sin y + \sin y + y\cos y) + \phi'(x) = e^x(x\,\sin y + \sin y + y\cos y)$, thus

$\phi'(x) = 0$ and $\phi(x) = c$, c is a constant which we can neglect, therefore $v = e^x(x\,\sin y + y\cos y)$. If

we put x = z and y = 0, thus $f(z) = z\, e^z$. $\hspace{3cm}$ (5 marks)

$\hspace{1cm}$ ii) From Inverse Lagrange formula, we get

$x = \dfrac{(y+7)(y-4)(y-47)(y-15)}{(3+7)(3-4)(3-47)(3-15)}(1) + \dfrac{(y-3)(y-4)(y-47)(y-15)}{(-7-3)(-7-4)(-7-47)(-7-15)}(5)$

$+ \dfrac{(y-3)(y+7)(y-47)(y-15)}{(4-3)(4+7)(4-47)(4-15)}(-13) + \dfrac{(y-3)(y+7)(y-4)(y-15)}{(47-3)(47+7)(47-4)(47-15)}(2).$

$+ \dfrac{(y-3)(y+7)(y-4)(y-47)}{(15-3)(15+7)(15-4)(15-47)}(-6)$

Put y = 0, so that we get one of the roots. $\hspace{3cm}$ (5 marks)

b- i) Put $Ln(2t) = -y$, thus $dt = -(e^{-y}/2)\,dy$, therefore $\displaystyle\int_0^{1/2} t^{m-1}\,\ln(1/2t)\,dt = \dfrac{1}{2^m}\int_0^{\infty} y e^{-my}\,dy$

$\hspace{1cm}$ Put my = x, hence $\displaystyle\int_0^{1/2} t^{m-1}\,\ln(1/2t)\,dt = \dfrac{1}{2^m}\int_0^{\infty} y e^{-my}\,dy = \dfrac{1}{2^m}\int_0^{\infty} \dfrac{x}{m} e^{-x}\,\dfrac{dx}{m} = \dfrac{1}{2^m m^2}$

ii) Since z = 5 is outside contour and z = 1 inside contour, therefore:

$$\oint_c \frac{\cos(z)}{z^2 - 6z + 5} \, dz = \oint_c \frac{\cos(z)/(z-5)}{z-1} \, dz = 2\pi i \, (\frac{\cos 1}{-4})$$

iii) $\int_2^\infty e^{-x^2 + 4x - 4} \, dx = \int_2^\infty e^{-(x-2)^2} \, dx$, Put $y = (x-2)^2 \Rightarrow dy = 2(x-2) \, dx \Rightarrow dx = -\frac{dy}{2\sqrt{y}}$

$$\int_2^\infty e^{-(x-2)^2} \, dx = \frac{1}{2} \int_0^\infty y^{-1/2} \, e^{-y} \, dy = \frac{\sqrt{\pi}}{2}$$

iv) $\oint_c \frac{z^3 + 5z + 7}{(z-i)^2} \, dz = = 2\pi i \, f'(i) = 4\pi i$

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